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LETTER TO THE EDITOR

Jordanian deformation of SL(2) as a contraction of its Drinfeld–Jimbo deformation

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Abstract. We introduce a contraction procedure for obtaining the Jordanian deformation of SL(2) from its Drinfeld–Jimbo deformation. To be specific, we obtain $\mathrm{SL}_h(2)$, its differential geometry, and its star product from their q-deformed counterparts. We also present a transformation of the generators of $\mathrm{U}_q(\mathrm{sl}(2))$ which, in a limit, yield the generators of $\mathrm{U}_h(\mathrm{sl}(2))$.

The group SL(2) admits two distinct quantizations: one is the well known Drinfeld–Jimbo (q-) deformation and the other is the so-called Jordanian (h-) deformation [1]. These quantum groups act on the q-plane, with the relation $xy = qyx$ and the h-plane, with the relation $xy - yx = hy^2$, respectively. It is shown that, up to isomorphisms, there exist only two quantum deformations of GL(2) which admit a central determinant [2]—the single-parameter q- and h-deformations. In addition, 2×2 quantum matrices admitting left and right quantum spaces are classified [3], i.e. the two parametric q- and h-deformations. In some vague sense it seems that the Jordanian deformation of GL(2) lies somewhat on the boundary of the Drinfeld–Jimbo deformation. In this letter we want to clarify explicitly the relation between these two deformations of SL(2).

For the q-deformation of SL(2) in particular (and also for SL(n)), one cannot develop a well developed differential geometry which simultaneously is bicovariant and has the correct number of generators and differential forms which satisfy the Leibnitz rule [4]. This is due to the fact that although the determinant of the quantum matrix is central to the algebra itself, it is not in the centre of the whole graded differential algebra. Therefore, one cannot form the quotient $\Omega(\mathrm{GL}_q(n)/\det(M_q(n)))$ to define $\Omega(\mathrm{SL}_q(n))$. It is the Jordanian deformation which successfully solves this difficulty [5]. In fact, if one insists on deforming not only the group itself but also its differential structure, the Jordanian deformation obtains an important success.

The idea of using the singular limits of transformations is not new. The contraction of Lie groups, as first introduced by Inonu and Wigner [6], is such a process. This contraction procedure has been successfully applied to quantum groups to obtain deformations of inhomogeneous groups such as $\mathrm{E}_q(2)$ and the Poincaré group [7–9].

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In this letter, we will show that $SL_h(2)$ [1, 2, 5, 10–15] can be obtained from $SL_q(2)$ by a singular limit of a similarity transformation. Not only the Hopf algebras but also the whole differential structure is obtained in this way. The same is also true for the inhomogeneous quantum group $IGL_h(2)$ which can be obtained from $IGL_q(2)$. In addition, the generators of the deformed universal enveloping algebra $U_h(sl(2))$ are related by a singular limit of a transformation to the generators of $U_q(sl(2))$.

To begin with we define

$$M' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in GL_q(2). \tag{1}$$

Throughout this letter we denote q-deformed objects by primed quantities. Unprimed quantities represent similarity transformed objects, which, in a certain limit, tend to h-deformed ones. $M' \in GL_q(2)$ means that the entries of M' fulfil the following commutation relations:

$$\begin{aligned} &a'c' = qc'a' \qquad b'd' = qd'b' \qquad [a', d'] = (q - q^{-1})b'c' \\ &a'b' = qb'a' \qquad c'd' = qd'c' \qquad c'b' = b'c'. \end{aligned} \tag{2}$$

Here [,] stands for the commutator. The quantum determinant D', defined below, lies in the centre of the algebra:

$$D' = \det_q M' := a'd' - qc'b'. \tag{3}$$

$GL_q(2)$ acts on the q-plane which is defined by $x'y' = qy'x'$. Now let us apply a change of basis in the coordinates of the q-plane by use of the following matrix:

$$g = \begin{pmatrix} 1 & \frac{h}{q-1} \\ 0 & 1 \end{pmatrix} \qquad \begin{aligned} x' &= x + \frac{h}{q-1}y \\ y' &= y. \end{aligned} \tag{4}$$

A simple calculation shows that the transformed generators x and y fulfil the relation $xy = qyx + hy^2$. The $q \to 1$ limit of this is exactly the commutation relation that defines the h-plane: $xy = yx + hy^2$. The transformation on $GL_q(2)$, corresponding to (4), is the following similarity transformation:

$$\begin{aligned} &M' = gMg^{-1} \\ &a' = a + \frac{h}{q-1}c \qquad b' = b + \frac{h}{q-1}(d-a) - \frac{h^2}{(q-1)^2}c \\ &c' = c \qquad d' = d - \frac{h}{q-1}c. \end{aligned} \tag{5}$$

It is clear that a change of basis in the quantum plane leads to the similarity transformation $M = g^{-1}M'g$ for the quantum group and the following similarity transformation for the corresponding R-matrix.

$$R = (g \otimes g)^{-1}R'(g \otimes g). \tag{6}$$

We use the following R-matrix for the q-deformation:

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & q & 1-q^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7}$$

For g given in (4), we obtain

$$\mathcal{R} := \lim_{q\to 1} R = \begin{pmatrix} 1 & -h & h & h^2 \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{8}$$

The algebra of functions $\mathrm{GL}_q(2)$ are obtained from the following relation

$$R'M'_1M'_2 = M'_2M'_1R'. \tag{9}$$

Applying transformations (5,6) one obtains in the limit

$$\mathcal{R}M_1M_2 = M_2M_1\mathcal{R}. \tag{10}$$

So, the entries of the transformed quantum matrix M fulfil the commutation relations of the $\mathrm{GL}_h(2)$.

$$\begin{aligned} &[a,c] = hc^2 \qquad [d,b] = h(D-d^2) \\ &[a,d] = hdc - hac \qquad [d,c] = hc^2 \\ &[b,c] = hac + hcd \qquad [b,a] = h(a^2 - D). \end{aligned} \tag{11}$$

Here the corresponding determinant is

$$D = \det_h M := ad - cb - hcd. \tag{12}$$

One can also obtain the above algebra by direct substitution of (5) in (2). Note that although the transformations (4), (5) are ill behaved as $q \to 1$ the resulting commutation relations are well defined in this limit. It is also important to note that this process cannot be reversed: one cannot use the inverse transformations to obtain $\mathrm{GL}_q(2)$ from $\mathrm{GL}_h(2)$. Because of the one-way nature of transformation, we call this process a *contraction*. It is easily shown that the co-unity, anitpode and co-product structures are also transformed to their h-deformed counterparts.

The inhomogeneous quantum group $\mathrm{IGL}_q(2)$ has two extra generators u' and v' which we arrange in matrix form

$$U' := \begin{pmatrix} u' \\ v' \end{pmatrix}. \tag{13}$$

The commutation relations for these extra generators, which correspond to translations, are:

$$\begin{aligned} &uv = qvu \\ &(av + uc) - q(cu + va) = 0 \\ &(bv + ud) - q(du + vb) = 0 \end{aligned} \tag{14}$$

Applying (4), (5) and $U' = gU$ in the limit $q \to 1$ we get

$$\begin{aligned} &uv - vu = hv^2 \\ &[b, v] + [u, d] = h\{d, v\} \\ &[a, v] + [u, c] = h\{c, v\} \end{aligned} \tag{15}$$

where $\{,\}$ stands for the anticommutator. These are the known commutation relations for $\mathrm{IGL}_h(2)$, [15].

A quantum group's differential structure is completely determined by its R-matrix [16, 17]. One therefore expects that by this similarity transformation the differential structure of the h-deformation [5] can be obtained from that of the q-deformation:

$$\begin{aligned} &M_2\,\mathrm{d}M_1 = R_{12}\,\mathrm{d}M_1 M_2 R_{21} \\ &\mathrm{d}M_2\,\mathrm{d}M_1 + R_{12}\,\mathrm{d}M_1\,\mathrm{d}M_2 R_{21} = 0 \end{aligned} \tag{16}$$

where $\mathrm{d}M = g^{-1}\,\mathrm{d}M' g$. Using the above relations the differential structure of $\mathrm{GL}_h(2)$ can easily be obtained from the corresponding differential structure of $\mathrm{GL}_q(2)$.

One can easily obtain two parametric deformation, $\mathrm{GL}_{hh'}(2)$, from $\mathrm{GL}_{qp}(2)$ by a similar procedure which results in the following map of the R-matrices:

$$R' = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & qp^2 & 1-q^2p^2 & 0 \\ 0 & 0 & q & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \to R = \begin{pmatrix} 1 & -h' & h' & hh' \\ 0 & 1 & 0 & -h \\ 0 & 0 & 1 & h \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

provided that, for $q \to 1$ and $p \to 1$, $(1 - qp^2)/(1 - q) \to h'/h$.

The star product of the elements of the functions on the group $\mathrm{SL}_q(2)$ is known. As far as we know for the Jordanian deformation of SL(2) these star products are not known. However, this method gives us the ability to obtain them. The star product of $\mathrm{SL}_q(2)$ is as below [18].

$$\begin{aligned} &a' * a' = a'^2 \qquad b' * b' = b'^2 \qquad c' * c' = c'^2 \qquad d' * d' = d'^2 \\ &a' * b' = qb' * a' = \sqrt{\frac{2q^2}{q^2+1}}a'b' \\ &a' * c' = qc' * a' = \sqrt{\frac{2q^2}{q^2+1}}a'c' \\ &b' * d' = qd' * b' = \sqrt{\frac{2q^2}{q^2+1}}b'd' \\ &c' * d' = qd' * c' = \sqrt{\frac{2q^2}{q^2+1}}c'd' \\ &b' * c' = c' * b' = \frac{2q}{q^2+1}b'c' \\ &a' * d' = d' * a' + (q - q^{-1})b' * c' = a'd' + \frac{q^2-1}{q^2+1}b'c'. \end{aligned} \tag{18}$$

Applying transformation (5) to these relations leads to the following star product for the Jordanian deformation of SL(2)

$$
\begin{aligned}
&a*a = a^2 + \frac{h^2}{4}c^2 \qquad a*b = ab + \frac{h}{2}(ad - bc - a^2) + \frac{h^2}{4}cd - \frac{h^3}{8}c^2 \\
&a*c = ac + \frac{h}{2}c^2 \qquad a*d = ad - \frac{h}{2}(ac - cd) - \frac{h^2}{4}c^2 \\
&c*d = cd - \frac{h}{2}c^2 \qquad b*b = b^2 + \frac{h^2}{4}[(a-d)^2 + 2bc] + \frac{h^4}{2}c^2 \\
&d*d = d^2 + \frac{h^2}{4}c^2 \qquad b*d = db - \frac{h}{2}(ad - bc - d^2) + \frac{h^2}{4}ac + \frac{h^3}{8}c^2 \\
&c*a = ac - \frac{h}{2}c^2 \qquad c*b = bc - \frac{h}{2}(ac + cd) + \frac{h^2}{4}c^2 \\
&c*c = c^2 \qquad b*a = ab - \frac{h}{2}(ad - bc - a^2) + \frac{h^2}{4}cd + \frac{h^3}{8}c^2 \\
&d*a = ad + \frac{h}{2}(ac - cd) - \frac{h^2}{4}c^2 \qquad d*b = bd + \frac{h}{2}(ad - bc - d^2) + \frac{h^2}{4}ac - \frac{h^3}{8}c^2 \\
&d*c = cd + \frac{h}{2}c^2 \qquad b*c = bc + \frac{h}{2}(ac + cd) + \frac{h^2}{4}c^2.
\end{aligned} \tag{19}
$$

To proceed with the deformed universal enveloping algebras we use the well known duality between $U_q(sl(2))$ and $SL_q(2)$. The algebra $U_q(sl(2))$ is generated by three generators $X^{\pm}$ and H. They can be arranged in the following matrix form:

$$
L'^{+} = \begin{pmatrix} q^{-\frac{1}{2}H} & (q - q^{-1})X^{+} \\ 0 & q^{\frac{1}{2}H} \end{pmatrix} \qquad L'^{-} = \begin{pmatrix} q^{\frac{1}{2}H} & 0 \\ (q^{-1} - q)X^{-} & q^{-\frac{1}{2}H} \end{pmatrix}. \tag{20}
$$

$U_q(sl(2))$ is generated by three generators $X^{\pm}$ and H. The duality relations are:

$$
\langle L'^{+}_{kl}, M'_{ij} \rangle = q^{\frac{1}{2}} R'^{+}_{ikjl} \langle L'^{-}_{kl}, M'_{ij} \rangle = q^{-\frac{1}{2}} R'^{-}_{ikjl} \tag{21}
$$

where

$$
R'^{+} := R'^{-1} \qquad R'^{-} := PR'P \qquad P_{ijmn} := \delta_{in}\delta_{jm}. \tag{22}
$$

Note that we have to use these because we started with an upper triangular R-matrix. Now we use g to get M, R^{+} and R^{-}.

$$
R^{\pm} = (g \otimes g)^{-1} R'^{\pm} (g \otimes g). \tag{23}
$$

Using these, it can be shown that

$$
\begin{aligned}
L^{+} &= g^{-1} L'^{+} g \\
L^{+} &= \begin{pmatrix} q^{-\frac{1}{2}H} & (q - q^{-1})X^{+} + \alpha(q^{\frac{1}{2}H} - q^{-\frac{1}{2}H}) \\ 0 & q^{\frac{1}{2}H} \end{pmatrix} \\
L^{-} &= g^{-1} L'^{-} g \\
L^{-} &= \begin{pmatrix} q^{\frac{1}{2}H} + \alpha(q^{-1} - q)X^{-} & -\alpha^2(q^{-1} - q)X^{-} + \alpha(q^{-\frac{1}{2}H} - q^{\frac{1}{2}H}) \\ (q^{-1} - q)X^{-} & q^{-\frac{1}{2}H} - \alpha(q^{-1} - q)X^{-} \end{pmatrix}
\end{aligned} \tag{24}
$$

where

$$\alpha = \frac{h}{q-1}. \tag{25}$$

Now we introduce the following generators:

$$\begin{aligned}
\mathcal{T} &:= q^{-H/2} \\
\mathcal{T}^{-1} &:= q^{H/2} \\
\mathcal{H} &:= \alpha h^{-1}(q^{H/2} - q^{-H/2}) + h^{-1}(q - q^{-1})X^{+} \\
\mathcal{Y} &:= \frac{2\alpha}{(q^{-1}-q)}(q^{H/2} - q^{-H/2}) - X^{+} - \frac{4\alpha h}{(q+1)(q^{-1}-q)}X^{-}.
\end{aligned} \tag{26}$$

The first three generators are the elements of L^+. The fourth one is built by a linear combination of the elements of L^+ and L^-, such that the resulting algebra of these generators remains non-singular in the limit $q \to 1$. In fact in this limit, both R^+ and R^- become upper triangular and one cannot obtain the commutation relations for the last generator. It can be shown that these generators fulfil the following set of commutation relations:

$$\begin{aligned}
& q\mathcal{T}\mathcal{H} - \mathcal{H}\mathcal{T} = 1 - \mathcal{T}^2 \\
& q\mathcal{H}\mathcal{T}^{-1} - \mathcal{T}^{-1}\mathcal{H} = \mathcal{T}^{-2} - 1 \\
& \mathcal{Y}\mathcal{T} - q^{-1}\mathcal{T}\mathcal{Y} = -\frac{h}{q+1}\{\mathcal{H}, \mathcal{T}\} \\
& \mathcal{Y}\mathcal{T}^{-1} - q\mathcal{T}^{-1}\mathcal{Y} = \frac{hq}{q+1}\{\mathcal{H}, \mathcal{T}^{-1}\} \\
& [\mathcal{H}, \mathcal{Y}] = -\frac{1}{q+1}\{Y, \mathcal{T}^{-1} + \mathcal{T}\}.
\end{aligned} \tag{27}$$

Setting $q = 1$ in these equations results to the known relations for $\mathrm{U}_h(\mathrm{sl}(2))$, [12]. It is easy to show that co-unity, co-product and antipode are also obtained in the $q \to 1$ limit from their corresponding q-deformed counterparts.

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References

[1] Demidov E E, Manin Yu I, Mukhin E E and Zhdanovich D V 1990 *Prog. Theor. Phys. Suppl.* **102** 203
[2] Kupershmidt B A 1992 *J. Phys. A: Math. Gen.* **25** L1239
[3] Ewen H, Ogievetsky O and Wess J 1991 *Lett. Math. Phys.* **22** 297
[4] Faddeev L D and Pyatov P N 1994 The differential calculus on quantum linear groups *Preprint* hep-th/9402070
[5] Karimipour V 1994 *Lett. Math. Phys.* **30** 87
[6] Inonu E and Wigner E P 1953 *Proc. Natl Acad. Sci. (USA)* **39** 510
[7] Celeghini E, Giachetti R, Sorace E and Tarlini M 1991 *J. Math. Phys.* **32** 1159
[8] Woronowicz S L 1992 *Commun. Math. Phys.* **149** 637
[9] Lukierski J, Nowicki A, Ruegg H and Tolstoy V N 1991 *Phys. Lett.* **264B** 331
Lukierski J and Nowicki A 1991 *Phys. Lett.* **279B** 299

[10] Dubois-Violette M and Launer G 1990 *Phys. Lett.* **245B** 175
[11] Zakrzewski S 1991 *Lett. Math. Phys.* **22** 287
[12] Ohn Ch 1992 *Lett. Math. Phys.* **25** 85
[13] Aghamohammadi A 1993 *Mod. Phys. Lett.* A **8** 2607
[14] Vladimirov A A 1993 *Mod. Phys. Lett.* A **8** 2573
[15] Aghamohammadi A, Rouhani S and Shariati A 1994 *J. Phys. A: Math. Gen.* **27** 7103
[16] Sudbery A 1992 *Phys. Lett.* **284B** 61
[17] Schupp P, Watts P and Zumino B 1992 *Lett. Math. Phys.* **25** 139
[18] Takhtajan L A 1990 *Introduction to Quantum Group and Integrable Massive Models of Quantum Field Theory* ed M L Ge and B H Zhao (Singapore: World Scientific)